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LETTER TO THE EDITOR

Universal substructures of percolation clusters: The skeleton

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Abstract. We define the 'skeleton' of a cluster aggregate as the set of all sites belonging to the shortest paths connecting a chosen site with the Lth chemical shell surrounding that site. The fractal properties of skeletons of percolation clusters at criticality have been studied, and we infer that the mass of the skeleton M_s scales with the chemical distance l (for $l \ll L$) as $M_s \sim l^{d_l^s}$, where $d_l^s = 1$ is universal for $1 \leq d \leq 6$. Numerical evidence which supports this conclusion is presented for $d = 2$, and an analytical proof is given for $d = 6$.

The investigation of the structure and substructure of cluster aggregates—especially the percolation cluster and its backbone—has been of great interest for some time (Skal and Shklovski 1975, Stanley 1977, Adler *et al* 1982, Mandelbrot 1982, Herrmann *et al* 1984). Recently the 'intrinsic dimension' d_l has been examined and relationships between d_l and other critical exponents have been established (Havlin and Nossal 1984, Havlin *et al* 1984a). The exponent d_l is defined by the relationship $M \sim l^{d_l}$, where M is the number of sites of a cluster within a chemical distance l units from an arbitrary point. The chemical distance between two points on the cluster is the shortest path of occupied sites linking those points.

In this letter we study the topological properties of substructures of percolation clusters and introduce the concept of the 'skeleton' of a cluster as such a substructure. The skeleton of a cluster aggregate is defined here as the ensemble of sites lying on the shortest paths connecting a chosen site, designated as the origin, with the Lth chemical shell defined with respect to that site. Consequently, all other sites of the cluster, those lying on 'dead ends' and longer paths, are excluded. Dead ends are branches which emanate from the skeleton but terminate before reaching the Lth shell. The structure and the 'intrinsic' dimension of the skeleton d_l, which relates the mass of the skeleton of a percolation cluster M_s to the chemical distance l through $M_s \sim l^{d_l^s}$ ($l \ll L$), are now studied. We argue that d_l^s is universal for $1 \leq d \leq 6$ and has the value $d_l^s = 1$. Numerical evidence is presented for $d = 2$ and exact results for $d = 6$.

The skeleton of a percolation cluster at $p = p_c$ in $d = 2$ can be generated by first using the cluster growth method of Alexandrowicz (1980) to grow a cluster of L shells on a triangular lattice (Havlin and Nossal 1984), and later discarding all sites which are not on the shortest paths to the Lth shell. Specifically, we determine the sites which belong to the skeleton by starting at the periphery (the Lth shell) of the cluster, and then searching for those occupied sites in shell $L-1$ which are connected to sites in the Lth shell. All other sites in shell $L-1$ are discarded. We then continue searching

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in a similar manner until the origin is reached. Using this method we generated skeletons of percolation clusters with $L = 100$, 200, 300, and 400 shells (see figure 1). The number of sites in the lth shell of a skeleton, $B^s(l)$, averaged over the configurations of the successful trials of attempts to generate clusters to a value of $l = L$, then was calculated. Results of these computations are shown in figure 2. As seen in figure 3, we find that the various curves can be superimposed if they are plotted as a function of l/L.

Some insight into this observed scaling may be obtained by examining the skeletons of percolation clusters in $d = 6$. Such clusters can be generated by percolation on a Cayley tree (de Gennes 1976). The fact that Cayley trees do not have loops simplifies our task. From studies on generalised branching processes it can be shown (Harris 1963) that, at criticality, the probability that any particular tree (with coordination

Figure 1. (*a*) Typical 2D percolation cluster on a triangular lattice, grown to a maximum chemcial size of $L = 400$. (*b*) The skeleton of the cluster shown in (*a*). Note the increasing number of branches as l increases. Although a few small loops are evident, their contribution to the total mass of the skeleton is negligible.

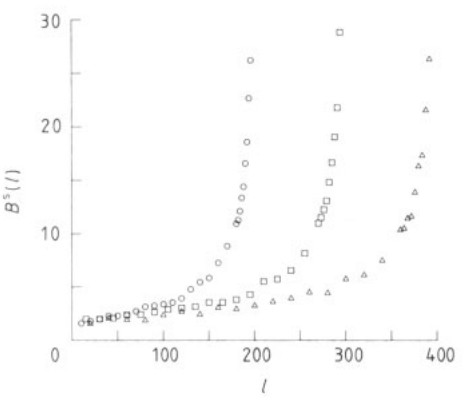

Figure 2. The number of sites $B^s(l)$ in the lth shell of the skeletons of percolation clusters which were grown to sizes of $L = 200$ (○), 300 (□), 400 (△) shells. (The number of configurations used in the averaging was: $L = 400$, $n = 16$; $L = 300$, $n = 23$; $L = 200$, $n = 33$; $L = 100$, $n = 63$ (see figure 3)).

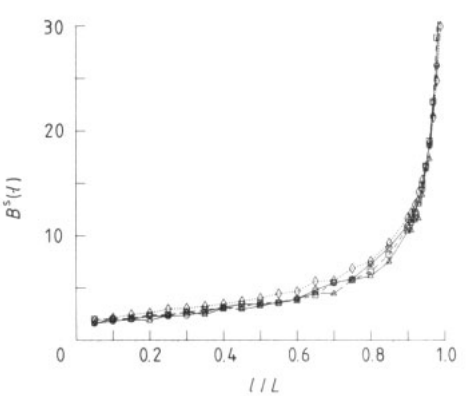

Figure 3. The data of figure 2, plotted as a function of l/L. ◇, $L = 100$; ○, $L = 200$; □, $L = 300$; △, $L = 400$.

number 3) grows to at least l shells is $P\{Z_l > 0\} \sim 4/l$ and that the expected number of occupied sites in the lth shell of a tree grown at criticality (given that the tree had not terminated before the lth step) is $B(l) \approx l/4$, for $l \gg 1$. Consequently, the expected number of sites in the lth shell of the *skeleton* of a tree grown to a maximum of L shells can be expressed as $B^s(l) \sim B(l) \cdot P\{Z_{L-l} > 0\}$, or $B^s(l) \sim (l/4) \cdot [4/(L-l)] \sim (l/L)/[1-(l/L)]$, for $1 \ll l \ll L$. A more careful asymptotic analysis shows that $B^s(l)$ contains additional terms, namely,

$$B^s(l) = B^s(0) + \left(\frac{\text{constant} + \frac{1}{4}[l + \ln(1 + l/4)]}{1 + \frac{1}{4}[(L-l) + \ln(1 + \frac{1}{4}(L-l))]} \right), \qquad 1 \ll l \tag{1a}$$

$$\approx B^s(0) + \frac{(l/L)^{d_l - 1}}{(1-l/L)^{d_l - 1}}, \qquad 1 \ll l < L \tag{1b}$$

where $d_l = 2$ and $B^s(0)$ is a constant. Thus, for $l \ll L$, $B^s(l) \approx B^s(0)$ and $M_s(l) \approx lB^s(0)$. Using the definition of d_l^s

$$M_s(l) \sim l^{d_l^s}, \tag{2}$$

we find $d_l^s = 1$ for $d = 6$. The $l = L$ limit of (1*a*) is $B(L) \sim L$, which is the result that was obtained in earlier studies (Havlin and Nossal 1984).

Note that although our derivation showed only that $B^s(l) \sim [(l/L)/(1-(l/L))]^\alpha$, where $\alpha = 1$, the exponent in (1*b*) has been written as $d_l - 1$. Thus we implicitly assume that (1) is valid for percolation clusters in all d. The data shown in figures 2 and 3, for $d = 2$, seem to support this inference. When we subtract a value $B^s(0) = \text{constant} = 1.44$ from the data for $L = 400$, and plot $\log[B^s(l) - B^s(0)]$ as a function of $\log[(l/L)/(1-(l/L))]$ as shown in figure 4, we find $\alpha = 0.67 \pm 0.03$, which is in close agreement with the value of the intrinsic dimension $d_l = 1.64$ which was found earlier (Havlin and Nossal 1984, Pike and Stanley 1981, Alexandrowicz 1980).

The fact that the data shown in figures 2-4 can be represented by setting $B^s(0) = \text{constant}$ (cf (1)) shows that, for $d = 2$,

$$M_s(l) \sim l^{d_l^s}, \qquad d_l^s = 1; \; L \to \infty. \tag{3}$$

This expression is in accordance with the theoretical results for $d = 6$. Since fractal exponents, usually do not decrease as d increases, we thus argue that $d_l^s = 1$ for all

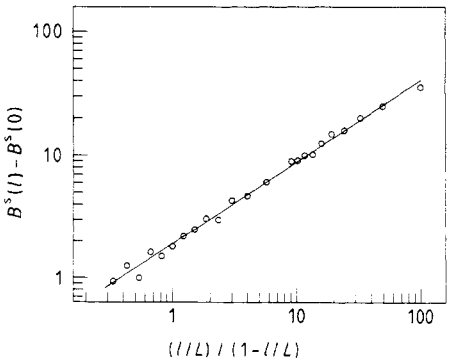

Figure 4. $B^s(l) - B^s(0)$, where $B^s(0) = 1.44$ is the average value for the first ten integer values of l, Plotted on logarithmic coordinates against $(l/L)/[1-(l/L)]$ for the $L = 400$ data shown in figure 2. The slope of the line is $d_l^s - 1 = 0.67$ (see (1)).

$1 \leqslant d \leqslant 6$. (The case $d = 1$ is trivial.) The interpretation of this result is that the skeletons of infinite percolation clusters essentially are chemically linear systems (Havlin *et al* 1984c). That is, over the range $l < 0.9L$ (i.e., except for values of l close to L, where a great deal of branching is evident), only a finite number of paths of minimum length connect any aribtrary point in a cluster with an arbitrary chemical shell surrounding that point. Thus, studies of the skeletons may suggest quantitative measures for the qualitative concept of finite ramification. It should be noted that the elastic backbone which has been studied by Hermann *et al* (1984) is the backbone of the skeleton defined in the present work.

The radius of gyration of the 2d skeletons also was calculated numerically as a function of l, i.e.,

$$R \sim l^{\tilde{\nu}_s} \sim R^{d_l^s/d_l^s}. \tag{4}$$

We found that the value of the exponent is $\nu_s = 0.87 \pm 0.02$, which is the same as the percolation result (Havlin and Nossal 1984), and we thus deduce

$$\tilde{\nu} = d_l/d_f = d_l^s/d_f^s = \tilde{\nu}_s. \tag{5}$$

The results of (4) and (5) are interesting because, since $d_l^s = 1$, it follows that

$$d_f^s = \tilde{\nu}^{-1} \tag{6}$$

as is the case for linear chains.

In conclusion, we assert that the intrinsic dimension of the skeletons of percolation clusters is universal for all d, namely, $d_l^s = 1$. This may substantiate the qualitative arguments that percolation clusters at criticality are finitely ramified. It is reasonable to assume that lattice animals also have the property $d_l^s = 1$ since they can be generated from percolation clusters when $p < p_c$ (Djordjevic *et al* 1984). Note, too, that for the case of the d-dimensional Sierpinski gasket it can be easily shown that $B^s(l) =$ constant $= d$ for $l < 0.5L$, i.e., again, $d_l^s = 1$. We presently are studying systems for which $d_l^s > 1$, and results will be presented elsewhere (Havlin *et al* 1984b).

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